Limited information and diversification in the growth optimal portfolio

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The Kelly game

1. in one turn, a fraction f of the current wealth can be invested
   - with the probability p, the invested amount is doubled
   - with the probability $1 - p$, the invested amount is lost
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- ***question:*** how to find the optimal investment fraction?
- ***well-known answer:*** maximise the exponential growth rate

$$G(f) := \left\langle \ln\left(1 + f\,R_1\right)\right\rangle$$

$R_1 =$ game return on one-turn basis

The Kelly game

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- in real life:
  - simultaneous games
  - unknown game properties
  - ...

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- the effective portfolio size m_{ef}

optimal portfolio constructed from ***N*** correlated assets	$\Longleftrightarrow$	optimal portfolio constructed from **???** uncorrelated assets

Effective portfolio size: properties

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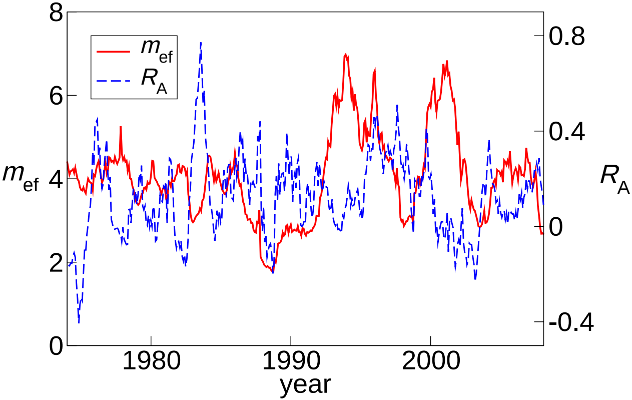
- all correlations identical:

$$m_{\text{ef}} = \frac{M}{1 + (M-1)C} < \frac{1}{C}$$

Effective portfolio size: evolution

20 current stocks from the DJIA (Jan 1973—Apr 2008)



Limited information: framework

- even “noisy” information in the form $p \pm \Delta$ is artificial
- let’s assume that we use only T past turns for learning

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our investment decision — $f^*(w, T)$

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- two interesting cases:

$$\lim_{T\to\infty} f^*(w,T) = 2\lim_{T\to\infty}\frac{w}{T} - 1 = 2p-1$$

$$f^*(T,T) = \frac{T}{T+2} < 1$$

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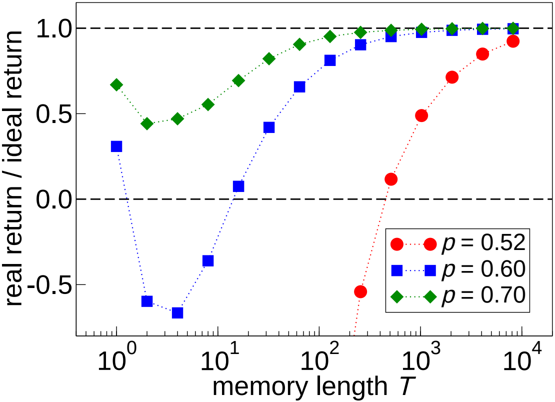
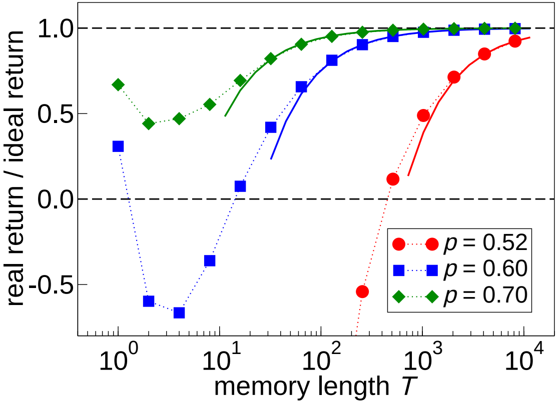


Limited information: results

$$G^*(p,T) \approx \underbrace{\ln 2 + p\ln p + (1-p)\ln(1-p)}_{\text{perfect information}} - \underbrace{1/(2T)}_{\text{limited information}}$$



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- "Great, I have my posterior $P(p|w, T)$ but what if…"
  - necessary because with enough data, prior beliefs are overruled!

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Thank you for your attention