Aging and heterogeneity in the growth of networks

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Growing networks

- Nodes and links are added with time
- Basic model: preferential attachment (PA)
  - Yule (1925), Simon (1955), Price (1976), Barabási & Albert (1999)
- Probability that a node acquires a new link is assumed proportional to the node's current degree

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- Pros: simplicity, resulting power-law degree distribution
- Cons: simplicity (deviations from the model observed in reality)

Deviations

- Many distributions claimed in the literature to be power laws fail in rigorous statistical tests (Clauset, Shalizi, Newman, 2009)
- Citation data shows patterns different from PA (Redner, 2005)
- No correlation between the age of a site and its number of incoming links in the WWW (Adamic & Huberman, 2000)
- A first-mover advantage in scientific citations exists but notable exceptions are present (Newman, 2009):
  “(There is) a hopeful sign that we as scientists do pay at least some attention to good papers that come along later”

Two generalizations of the basic PA

- Fitness model (Bianconi & Barabási, 2001):
  - Each node has fitness that influences the attachment probability

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  - Fitness distribution with unbounded support $\implies$ link condensation
- Aging of sites (Dorogovtsev & Mendes, 2000):
  - For a node that appeared at time s, the attachment rate is

$$P(i,t) \sim k_i(t)/(t-s)^\alpha$$

  - Scale-free $P(k)$ is observed only for very slow decay ($\alpha < 1$)

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- But is this really relevant?

Empirical evidence

- Citation data provided by the American Physical Society
  - 450’084 papers published by the APS from 1893 to 2009
  - 4’691’938 citations within the APS journals
- In-degree distribution:
  - $\alpha = 2.29 \pm 0.01$, $x_{\text{min}} = 50$
  - Statistical significance only for $x_{\text{min}} \gtrsim 150$
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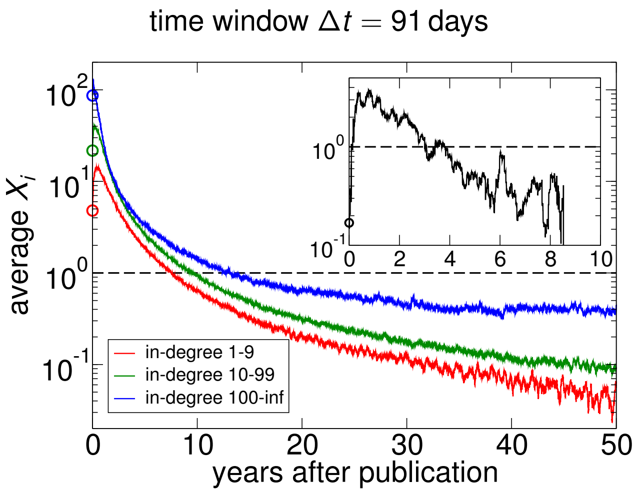
  - Log-normal distribution does not fit the data better
- Empirical relevance of paper i at time t: $X_i(t, \Delta t)$

$$\Delta k_i(t, \Delta t) = C(t, \Delta t) \frac{k_i(t)}{\sum_j k_j(t)} \times X_i(t, \Delta t)$$

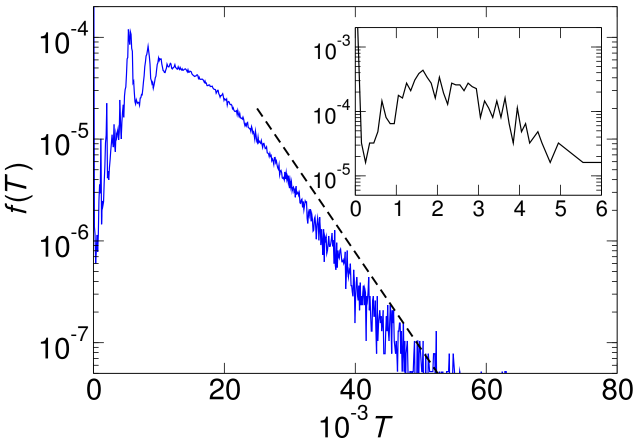
  - Δt is the time window, $C(t, \Delta t)$ is the number of new citations

Decay of relevance in the APS data



Heterogeneity of total relevance in the APS data

$$T_i := \sum_t X_i(t)$$



Solving the model

$$P(i,t) = \frac{k_i(t)R_i(t)}{\sum_{j=1}^{t} k_j(t)R_j(t)} = \frac{k_i(t)R_i(t)}{\Omega(t)}$$

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$$\Downarrow$$

$$\langle k_i^F\rangle = \exp\left(\frac{1}{\Omega^*}\int_0^\infty R_i(t)\,\mathrm{d}t\right) = \exp\left(T_i/\Omega^*\right)$$

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- $\int_0^\infty R_i(t)\,\mathrm{d}t \overset{!}{<} \infty$ to observe growth saturation and $\Omega(t) \to \Omega^*$
- The form of $R(t)$ matters little – it's T what's important
- Ω^* determined by self-consistency: the average degree is two

$$\int \varrho(T)\,\mathrm{e}^{T/\Omega^*}\mathrm{d}T = 2 \qquad (\varrho(T) \implies \Omega^*)$$

Degree distributions

- One can show that $f(k|T)$ is a narrow distribution
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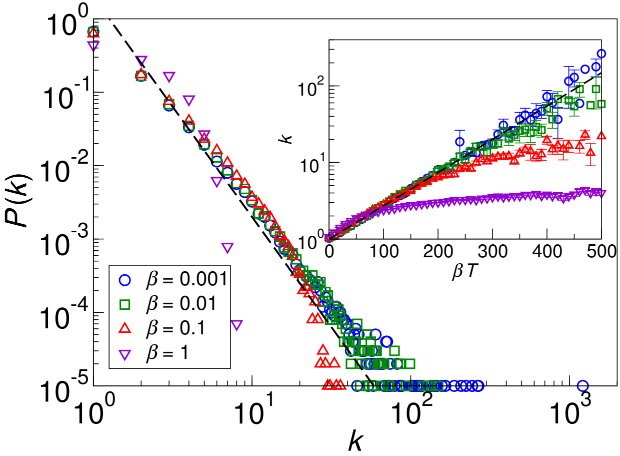
- Some examples:
  - $\varrho(T)$ normally distributed $\implies$ log-normal $f(k)$
  - $\varrho(T)$ exponentially distributed $\implies$ power-law $f(k)$
  - $\varrho(T) = \alpha e^{-\alpha T} \implies f(k) \sim k^{-3}$ (exactly as for PA!)

Numerical results

$R_i(t) = R_i(0)e^{-\beta(t-t_i)}$, $R_i(0)$ exponentially distributed



Open questions

- Study clustering coefficient and degree correlations
- Directed nature of the citation network
- Accelerating growth of the network
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- Why $\varrho(T)$ for citation data shows an exponential tail?
- What about other systems where PA is at work?

Challenges

- Mitzenmacher (2005): types of results when studying power laws
  1. *Observe*: Gather data and demonstrate a power law fit
  2. *Interpret*: Explain the significance of the power law behavior
  3. *Model*: Propose an underlying model that explains it
  4. *Validate*: Find data to validate/modify the model
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- Ad 4: Maximum Likelihood Estimation can help fit individual relevance values
- Ad 5: Knowledge of the dynamics can help select the (currently) most relevant nodes

Thank you for your attention