Breakdown of the mean-field approximation in a wealth distribution model

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International Workshop on Challenges and Visions in the Social Sciences

Introduction

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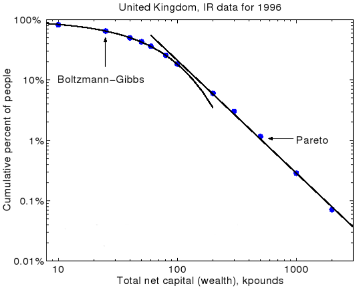
- 1st empirical observation: Vilfredo Pareto, 1897
- wealth and income also follow power-law distributions



(Drăgulescu, Yakovenko, 2001)

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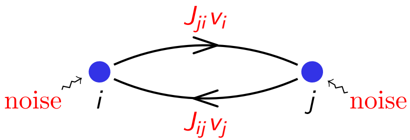

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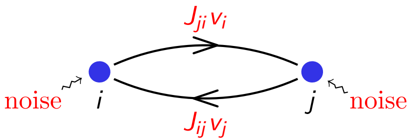

- simplest case: $J_{ij} = 1/(N-1)$

$$\mathrm{d}v_i(t) = \underbrace{\Big(\tilde{v}_i(t) - v_i(t)\Big)\,\mathrm{d}t}_{\text{trading}} + \underbrace{\sqrt{2}\sigma v_i(t)\,\mathrm{d}W_i(t)}_{\text{speculations}}$$

Mean-field solution

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Fokker-Planck equation for the wealth distribution $f(v_i)$

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stationary solution

$$f(v_i) = C\,\exp\left[-\frac{1}{\sigma^2 v_i}\right] v_i^{-2-1/\sigma^2}$$

oh, power-law tail!

Something is wrong. . .

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Something is wrong...

- this stationary solution has the fixed variance $\frac{\sigma^2}{1-\sigma^2}$ $(\sigma < 1)$
- by summing all dv_i's we get

$$\mathrm{d}v_T = \sqrt{2}\sigma \sum_{i=1}^{N} v_i \, \mathrm{d}W_i$$

- such dispersion never stops...

What happens for $N < \infty$?

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- without the mean-field approximation, the Fokker-Planck equation cannot be solved
- **the key idea:** let's take a look at averages!
- with the Itô lemma and average $\langle \cdot \rangle$ over noise

$$
\begin{aligned}
\frac{\mathrm{d}\langle v_i^2 \rangle}{\mathrm{d}t} &= 2\left[\langle v_i v_j \rangle - (1-\sigma^2)\langle v_i^2 \rangle\right] \\
\frac{\mathrm{d}\langle v_i v_j \rangle}{\mathrm{d}t} &= \frac{2}{N-1}\left[\langle v_i^2 \rangle - \langle v_i v_j \rangle\right] \\
\langle v_i v_j \rangle(0) &= 1 \\
\langle v_i^2 \rangle(0) &= 1
\end{aligned}
$$

Time scales

- from $\langle v_i v_j \rangle$ and $\langle v_i^2 \rangle$ we obtain $\text{var}[v_i]$ and C_{ij}

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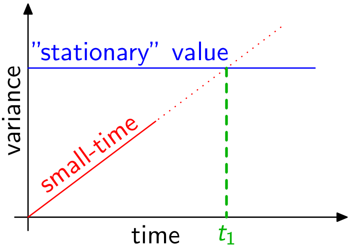

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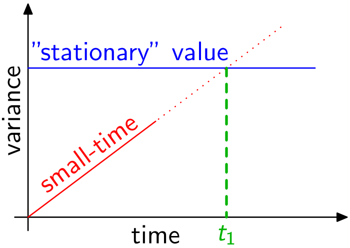

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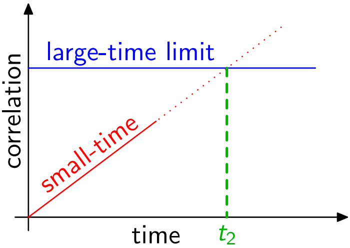

$$t_2 = (1 - \sigma^2)\, N$$

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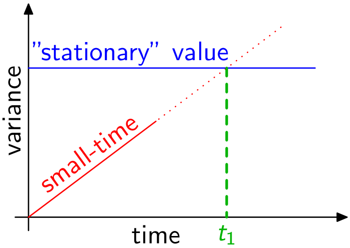
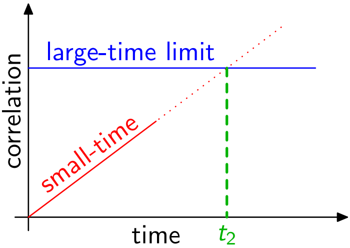




$$t_1 = \frac{1}{2(1-\sigma^2)}$$

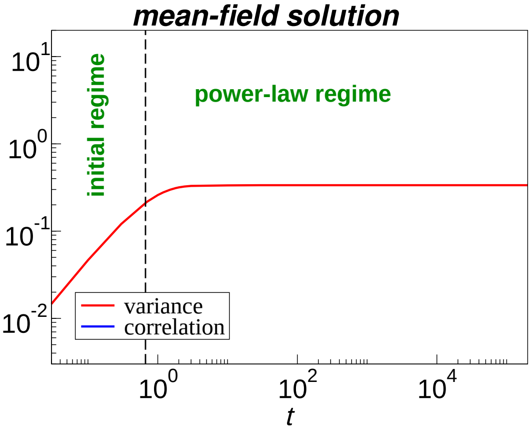
$$t_2 = (1-\sigma^2)\,N$$

- variance contains a term proportional to $\exp[\lambda t]$, $\lambda > 0$

$$t_3 = 1/\lambda = \frac{1-\sigma^2}{2\sigma^2} N + O(1)$$

Time evolution

$N = 10^4$, $\sigma^2 = 0.5$



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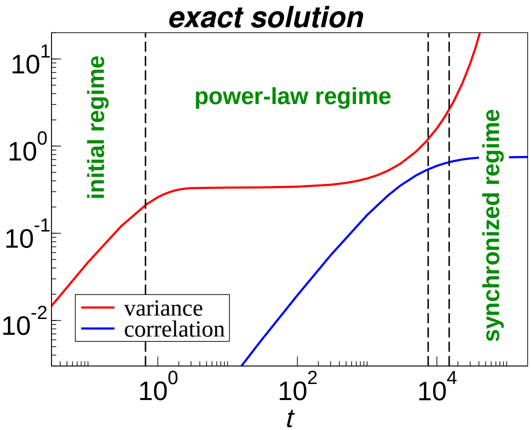

Concluding remarks

- for any $N < \infty$, there is no stationary wealth distribution
- when the average number of neighbours is z, correlations appear in time $O(z)$
- can taxes stabilize the system?
- what is the wealth distribution in the synchronizaed regime?
- the case $\sigma > 1$ must be treated separately
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thank you for your attention