Diversification and limited information in the Kelly game

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The Kelly game

1. in one turn, a fraction f of the current wealth can be invested
   - with the probability p, the invested amount is doubled
   - with the probability $1 - p$, the invested amount is lost
2. repeat (infinitely) many times
3. winning probability p is constant and known

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- ***question:*** how to find the optimal investment fraction?
- ***well-known answer:*** maximise the exponential growth rate

$$G(f) := \left\langle \ln\left(1 + f R_1\right) \right\rangle$$

$R_1 =$ game return on one-turn basis

The Kelly game

- optimal investment fraction

$$f^*(p) = \begin{cases} 0 & p \in [0; \frac{1}{2}] \\ 2p - 1 & p \in (\frac{1}{2}; 1] \end{cases}$$

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- in real life:
  - simultaneous games
  - unknown game properties
  - ...

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Insider vs outsider: intro

- M games simultaneously played
- ***insider strategy***: know one game better
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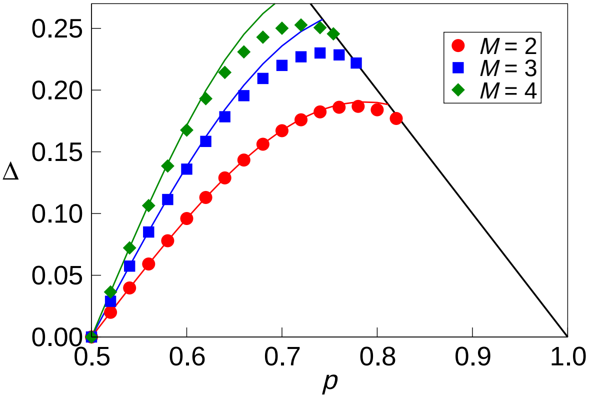
- ***insider strategy***: know one game better
- ***outsider strategy***: gain by diversification
- when the outsider outperforms the insider?
  1. little insider's information
  2. extensive outsider's diversification

Insider vs outsider: framework

1. M games
2. the winning probability of each game is either $p - \Delta$ or $p + \Delta$ (changing each turn randomly)
3. insider knows the exact winning probability for one game (no diversification)
4. outsider knows only the average winning probability p (invests evenly in M games)

Insider vs outsider: results

first approximation: $\Delta \approx \left(p - \frac{1}{2}\right)\left(\sqrt{2M} - 1\right)$



Limited information: framework

- even “noisy” information in the form $p \pm \Delta$ is artificial
- let’s assume that we use only T past turns for learning

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information about the game — $\varrho(p|w, T)$

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our investment decision — $f^*(w, T)$

Limited information: derivation

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- observing w wins in T turns gives us the information

$$\varrho(p|w,T) \propto \pi(p) P(w|p,T)$$

  - here $P(w|p,T)$ is the binomial distribution

$$P(w|p,T) = \binom{T}{w} p^w (1-p)^{T-w}$$

  - $\pi(p)$ is the prior distribution of p

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- the optimal investment fraction is

$$f^*(w, T) = \begin{cases} 0 & w \leq T/2 \\ \dfrac{2w - T}{T + 2} & w > T/2 \end{cases}$$

- two interesting cases:

$$\lim_{T\to\infty} f^*(w, T) = 2 \lim_{T\to\infty} \frac{w}{T} - 1 = 2p - 1$$

$$f^*(T, T) = \frac{T}{T + 2} < 1$$

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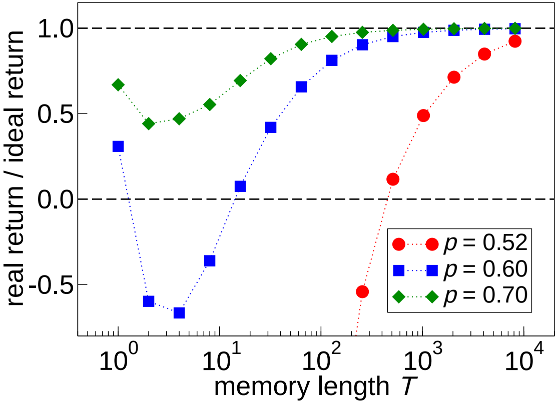
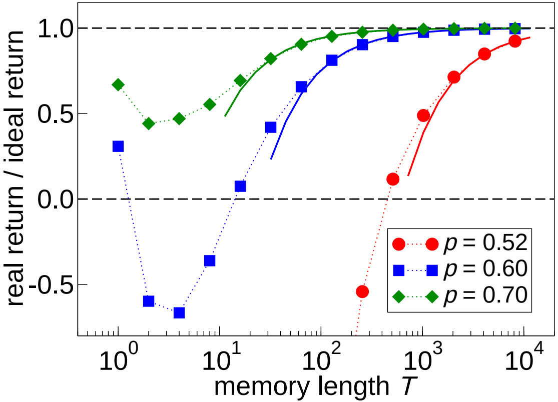


Limited information: results

$$G^*(p,T) \approx \underbrace{\ln 2 + p\ln p + (1-p)\ln(1-p)}_{\text{perfect information}} - \underbrace{1/(2T)}_{\text{limited information}}$$



The role of prior information

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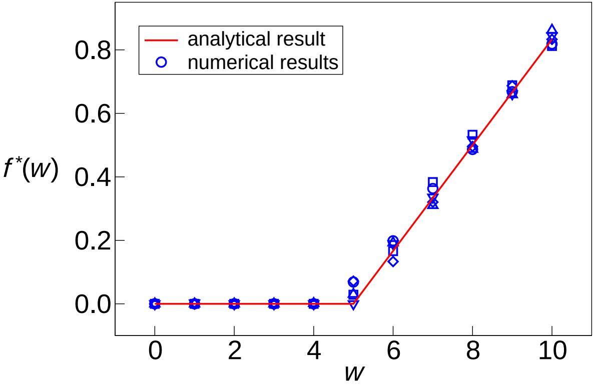
  2. aggregate information about p evolving in time



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- perfect information: return **16.7%**

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- our framework is too simple to allow for more realistic considerations...

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Thank you for your attention