27. Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$.

28. Cardioid overlapping a circle Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

29. One leaf of a rose Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.

30. Snail shell Find the area of the region enclosed by the positive x-axis and spiral $r = 4\theta/3, 0 \le \theta \le 2\pi$. The region looks like a snail shell.

31. Cardioid in the first quadrant Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

32. Overlapping cardioids Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

27. $\int_0^{\pi/2}\int_0^{2\sqrt{2-\sin 2\theta}} r\,dr\,d\theta = 2\int_0^{\pi/2}(2-\sin 2\theta)\,d\theta = 2(\pi-1)$

28. $A = 2\int_0^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left(2\cos\theta+\cos^2\theta\right)d\theta = \frac{8+\pi}{4}$

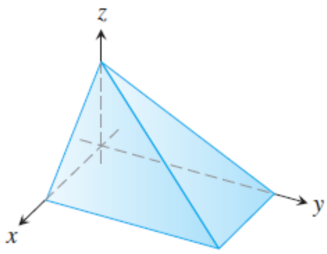
29. $A = 2\int_0^{\pi/6}\int_0^{12\cos 3\theta} r\,dr\,d\theta = 144\int_0^{\pi/6}\cos^2 3\theta\,d\theta = 12\pi$

30. $A = \int_0^{2\pi}\int_0^{4\theta/3} r\,dr\,d\theta = \frac{8}{9}\int_0^{2\pi}\theta^2\,d\theta = \frac{64\pi^3}{27}$

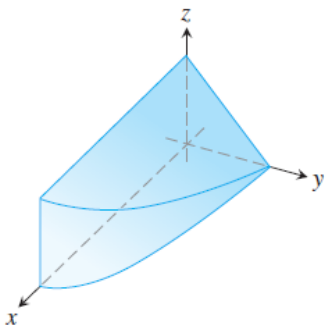
31. $A = \int_0^{\pi/2}\int_0^{1+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3}{2}+2\sin\theta-\frac{\cos 2\theta}{2}\right)d\theta = \frac{3\pi}{8}+1$

32. $A = 4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/2}\left(\frac{3}{2}-2\cos\theta+\frac{\cos 2\theta}{2}\right)d\theta = \frac{3\pi}{2}-4$

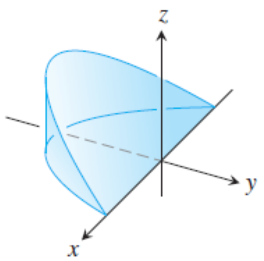
24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$



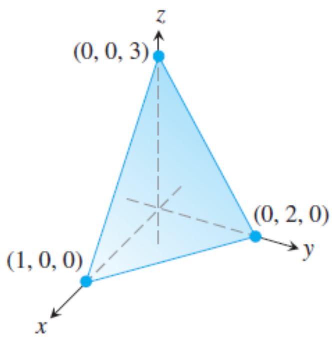
25. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



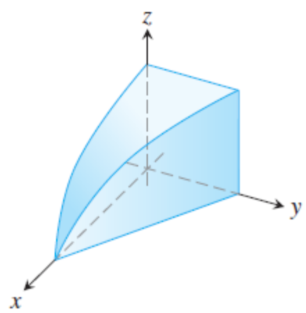
26. The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$



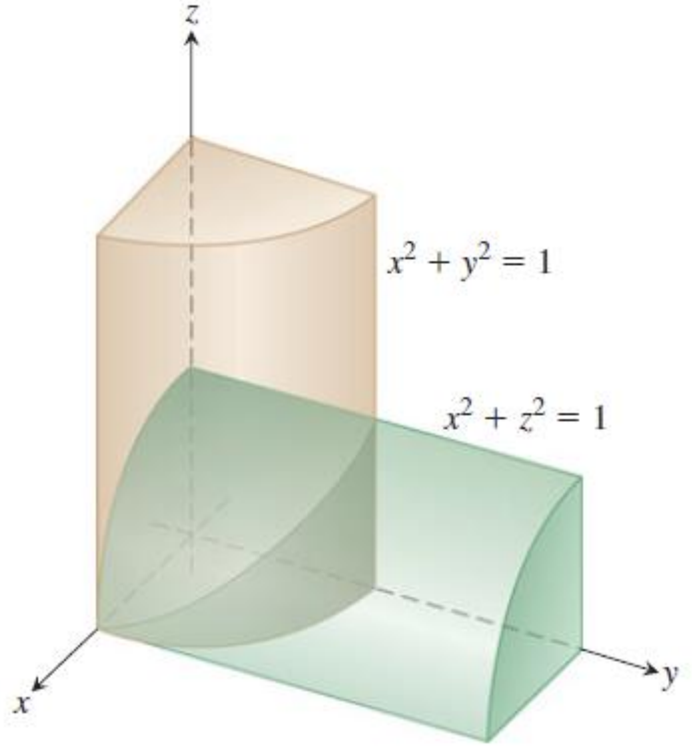
27. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through (1, 0, 0), (0, 2, 0), and (0, 0, 3)



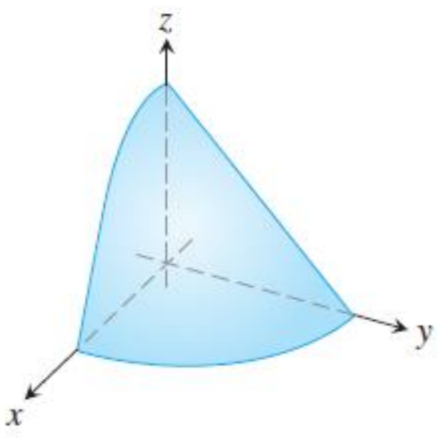
28. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$, and the surface $z = \cos(\pi x/2)$, $0 \le x \le 1$



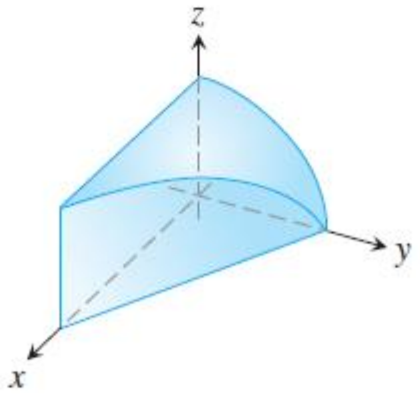
29. The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown in the accompanying figure



30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$



31. The region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$



25. $V=\int_0^4\int_0^{\sqrt{4-x}}\int_0^{2-y} dz\,dy\,dx=\int_0^4\int_0^{\sqrt{4-x}}(2-y)\,dy\,dx=\int_0^4\left[2\sqrt{4-x}-\left(\frac{4-x}{2}\right)\right]dx=\left[-\frac{4}{3}(4-x)^{3/2}+\frac{1}{4}(4-x)^2\right]_0^4$
$=\frac{4}{3}(4)^{3/2}-\frac{1}{4}(16)=\frac{32}{3}-4=\frac{20}{3}$

26. $V=2\int_0^1\int_{-\sqrt{1-x^2}}^0\int_0^{-y} dz\,dy\,dx=-2\int_0^1\int_{-\sqrt{1-x^2}}^0 y\,dy\,dx=\int_0^1\left(1-x^2\right)dx=\frac{2}{3}$

27. $V=\int_0^1\int_0^{2-2x}\int_0^{3-3x-3y/2} dz\,dy\,dx=\int_0^1\int_0^{2-2x}\left(3-3x-\frac{3}{2}y\right)dy\,dx=\int_0^1\left[6(1-x)^2-\frac{3}{4}\cdot 4(1-x)^2\right]dx$
$=\int_0^1 3(1-x)^2\,dx=\left[-(1-x)^3\right]_0^1=1$

28. $V=\int_0^1\int_0^{1-x}\int_0^{\cos(\pi x/2)} dz\,dy\,dx=\int_0^1\int_0^{1-x}\cos\left(\frac{\pi x}{2}\right)dy\,dx=\int_0^1\left(\cos\frac{\pi x}{2}\right)(1-x)\,dx$
$=\int_0^1\cos\left(\frac{\pi x}{2}\right)dx-\int_0^1 x\cos\left(\frac{\pi x}{2}\right)dx=\left[\frac{2}{\pi}\sin\frac{\pi x}{2}\right]_0^1-\frac{4}{\pi^2}\int_0^{\pi/2}u\cos u\,du=\frac{2}{\pi}-\frac{4}{\pi^2}\left[\cos u+u\sin u\right]_0^{\pi/2}$
$=\frac{2}{\pi}-\frac{4}{\pi^2}\left(\frac{\pi}{2}-1\right)=\frac{4}{\pi^2}$

29. $V=8\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx=8\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2}\,dy\,dx=8\int_0^1\left(1-x^2\right)dx=\frac{16}{3}$

30. $V=\int_0^2\int_0^{4-x^2}\int_0^{4-x^2-y} dz\,dy\,dx=\int_0^2\int_0^{4-x^2}\left(4-x^2-y\right)dy\,dx=\int_0^2\left[\left(4-x^2\right)^2-\frac{1}{2}\left(4-x^2\right)^2\right]dx$
$=\frac{1}{2}\int_0^2\left(4-x^2\right)^2dx=\int_0^2\left(8-4x^2+\frac{x^4}{2}\right)dx=\frac{128}{15}$

31. $V=\int_0^4\int_0^{\sqrt{16-y^2}/2}\int_0^{4-y} dx\,dz\,dy=\int_0^4\int_0^{\sqrt{16-y^2}/2}(4-y)\,dz\,dy=\int_0^4\frac{\sqrt{16-y^2}}{2}(4-y)\,dy$
$=\int_0^4 2\sqrt{16-y^2}\,dy-\frac{1}{2}\int_0^4 y\sqrt{16-y^2}\,dy=\left[y\sqrt{16-y^2}+16\sin^{-1}\frac{y}{4}\right]_0^4+\left[\frac{1}{6}\left(16-y^2\right)^{3/2}\right]_0^4$
$=16\left(\frac{\pi}{2}\right)-\frac{1}{6}(16)^{3/2}=8\pi-\frac{32}{3}$

67. Center of mass A solid of constant density is bounded below by the plane $z=0$, above by the cone $z=r$, $r\geq 0$, and on the sides by the cylinder $r=1$. Find the center of mass.

68. Centroid Find the centroid of the region in the first octant that is bounded above by the cone $z=\sqrt{x^2+y^2}$, below by the plane $z=0$, and on the sides by the cylinder $x^2+y^2=4$ and the planes $x=0$ and $y=0$.

67. $M = 4\int_0^{\pi/2}\int_0^1\int_0^r dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 r^2\, dr\, d\theta = \frac{4}{3}\int_0^{\pi/2} d\theta = \frac{2\pi}{3}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_0^r z\, dz\, r\, dr\, d\theta$
$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^3 dr\, d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4} \Rightarrow \bar{z} = \frac{M_{xy}}{M} = \left(\frac{\pi}{4}\right)\left(\frac{3}{2\pi}\right) = \frac{3}{8}$, and $\bar{x} = \bar{y} = 0$, by symmetry

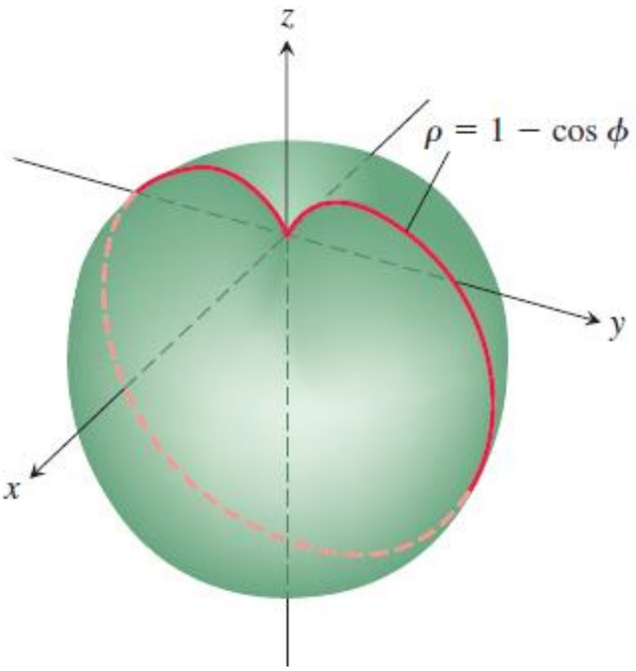
68. $M = \int_0^{\pi/2}\int_0^2\int_0^r dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 r^2 dr\, d\theta = \frac{8}{3}\int_0^{\pi/2} d\theta = \frac{4\pi}{3}$; $M_{yz} = \int_0^{\pi/2}\int_0^2\int_0^r x\, dz\, r\, dr\, d\theta$
$= \int_0^{\pi/2}\int_0^2 r^3\cos\theta\, dr\, d\theta = 4\int_0^{\pi/2}\cos\theta\, d\theta = 4$; $M_{xz} = \int_0^{\pi/2}\int_0^2\int_0^r y\, dz\, r\, dr\, d\theta = \int_0^{\pi/2}\int_0^2 r^3\sin\theta\, dr\, d\theta$
$= 4\int_0^{\pi/2}\sin\theta\, d\theta = 4$; $M_{xy} = \int_0^{\pi/2}\int_0^2\int_0^r z\, dz\, r\, dr\, d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^2 r^3\, dr\, d\theta = 2\int_0^{\pi/2} d\theta = \pi \Rightarrow \bar{x} = \frac{M_{yz}}{M} = \frac{3}{\pi}$,
$\bar{y} = \frac{M_{xz}}{M} = \frac{3}{\pi}$, and $\bar{z} = \frac{M_{xy}}{M} = \frac{3}{4}$

82. Mass of planet's atmosphere A spherical planet of radius R has an atmosphere whose density is $\mu = \mu_0 e^{-ch}$, where h is the altitude above the surface of the planet, μ_0 is the density at sea level, and c is a positive constant. Find the mass of the planet's atmosphere.

82. The mass of the planet's atmosphere to an altitude h above the surface of the planet is the triple integral
$M(h) = \int_0^{2\pi}\int_0^{\pi}\int_R^h \mu_0 e^{-c(\rho-R)}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_R^h\int_0^{2\pi}\int_0^{\pi}\mu_0 e^{-c(\rho-R)}\rho^2\sin\phi\, d\phi\, d\theta\, d\rho$
$= \int_R^h\int_0^{2\pi}\left[\mu_0 e^{-c(\rho-R)}\rho^2(-\cos\phi)\right]_0^{\pi} d\theta\, d\rho = 2\int_R^h\int_0^{2\pi}\mu_0 e^{cR}e^{-c\rho}\rho^2 d\theta\, d\rho = 4\pi\mu_0 e^{cR}\int_R^h e^{-c\rho}\rho^2 d\rho$
$= 4\pi\mu_0 e^{cR}\left[-\frac{\rho^2 e^{-c\rho}}{c} - \frac{2\rho e^{-c\rho}}{c^2} - \frac{2e^{-c\rho}}{c^3}\right]_R^h$ (by parts)
$= 4\pi\mu_0 e^{cR}\left(-\frac{h^2e^{-ch}}{c} - \frac{2he^{-ch}}{c^2} - \frac{2e^{-ch}}{c^3} + \frac{R^2e^{-cR}}{c} + \frac{2Re^{-cR}}{c^2} + \frac{2e^{-cR}}{c^3}\right)$.

The mass of the planet's atmosphere is therefore $M = \lim_{h\to\infty} M(h) = 4\pi\mu_0\left(\frac{R^2}{c} + \frac{2R}{c^2} + \frac{2}{c^3}\right)$.

40. Moment of inertia of an apple Find the moment of inertia about the z-axis of a solid of density $\delta = 1$ enclosed by the spherical coordinate surface $\rho = 1 - \cos\phi$. The solid is the red curve rotated about the z-axis in the accompanying figure.



40. $I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\theta} (\rho\sin\phi)^2\left(\rho^2\sin\phi\right) d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\theta} \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta$

$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} (1-\cos\phi)^5 \sin^3\phi\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} (1-\cos\phi)^6 (1+\cos\phi)\sin\phi\, d\phi\, d\theta;\ \begin{bmatrix} u = 1-\cos\phi \\ du = \sin\phi\, d\phi \end{bmatrix}$

$\rightarrow \frac{1}{5}\int_0^{2\pi}\int_0^{2} u^6(2-u)\, du\, d\theta = \frac{1}{5}\int_0^{2\pi}\left[\frac{2u^7}{7} - \frac{u^8}{8}\right]_0^2 d\theta = \frac{1}{5}\int_0^{2\pi}\left(\frac{1}{7}-\frac{1}{8}\right)2^8\, d\theta = \frac{1}{5}\int_0^{2\pi}\frac{2^3\cdot 2^5}{56}\, d\theta = \frac{32}{35}\int_0^{2\pi} d\theta = \frac{64\pi}{35}$

14. Use the transformation $x = u + (1/2)v$, $y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x - y)e^{(2x-y)^2}\, dx\, dy$$

by first writing it as an integral over a region G in the uv-plane.

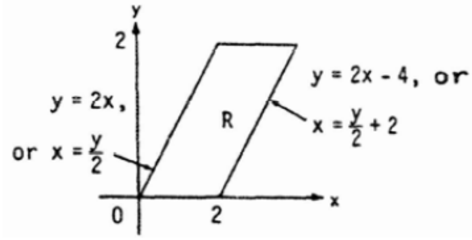
15. Use the transformation $x = u/v$, $y = uv$ to evaluate the integral sum

$$\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\, dx\, dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\, dx\, dy.$$

14. $x = u + \frac{v}{2}$ and $y = v \Rightarrow 2x - y = (2u + v) - v = 2u$

and $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = 1$; next,

$u = x - \frac{v}{2} = x - \frac{y}{2}$ and $v = y$, so the boundaries of the region of integration R in the xy-plane are transformed to the boundaries of G:



xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = \frac{y}{2}$	$u + \frac{v}{2} = \frac{v}{2}$	$u = 0$
$x = \frac{y}{2} + 2$	$u + \frac{v}{2} = \frac{v}{2} + 2$	$u = 2$
$y = 0$	$v = 0$	$v = 0$
$y = 2$	$v = 2$	$v = 2$

$\Rightarrow \int_0^2 \int_{y/2}^{(y/2)+2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy = \int_0^2 \int_0^2 v^3(2u)e^{4u^2}\,du\,dv = \int_0^2 v^3\left[\frac{1}{4}e^{4u^2}\right]_0^2 dv = \frac{1}{4}\int_0^2 v^3\left(e^{16}-1\right)dv$

$= \frac{1}{4}\left(e^{16}-1\right)\left[\frac{v^4}{4}\right]_0^2 = e^{16} - 1$

15. $x = \frac{u}{v}$ and $y = uv \Rightarrow \frac{y}{x} = v^2$ and $xy = u^2$; $\frac{\partial(x,y)}{\partial(u,v)} = J(u,v) = \begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$;

$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v = 1$, and $y = 4x \Rightarrow v = 2$; $xy = 1 \Rightarrow u = 1$, and $xy = 4 \Rightarrow u = 2$; thus

$\int_1^2 \int_{1/y}^{y} \left(x^2 + y^2\right) dx\,dy + \int_2^4 \int_{y/4}^{4/y} \left(x^2+y^2\right) dx\,dy = \int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2v^2\right)\left(\frac{2u}{v}\right) du\,dv = \int_1^2 \int_1^2 \left(\frac{2u^3}{v^3} + 2u^3v\right) du\,dv$

$= \int_1^2 \left[\frac{u^4}{2v^3} + \frac{1}{2}u^4v\right]_1^2 dv = \int_1^2 \left(\frac{15}{2v^3} + \frac{15v}{2}\right) dv = \left[-\frac{15}{4v^2} + \frac{15v^2}{4}\right]_1^2 = \frac{225}{16}$

35. Mass of wire with variable density Find the mass of a thin wire lying along the curve $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}$, $0 \le t \le 1$, if the density is **(a)** $\delta = 3t$ and **(b)** $\delta = 1$.

36. Center of mass of wire with variable density Find the center of mass of a thin wire lying along the curve $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + (2/3)t^{3/2}\mathbf{k}$, $0 \le t \le 2$, if the density is $\delta = 3\sqrt{5 + t}$.

37. Moment of inertia of wire hoop A circular wire hoop of constant density δ lies along the circle $x^2 + y^2 = a^2$ in the xy-plane. Find the hoop's moment of inertia about the z-axis.

38. Inertia of a slender rod A slender rod of constant density lies along the line segment $\mathbf{r}(t) = t\mathbf{j} + (2 - 2t)\mathbf{k}$, $0 \le t \le 1$, in the yz-plane. Find the moments of inertia of the rod about the three coordinate axes.

35. $\mathbf{r}(t)=\sqrt{2}t\mathbf{i}+\sqrt{2}t\mathbf{j}+\left(4-t^2\right)\mathbf{k},\ 0\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=\sqrt{2}\mathbf{i}+\sqrt{2}\mathbf{j}-2t\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{2+2+4t^2}=2\sqrt{1+t^2};$

(a) $M=\int_C \delta\, ds=\int_0^1 (3t)\left(2\sqrt{1+t^2}\right)dt=\left[2\left(1+t^2\right)^{3/2}\right]_0^1=2\left(2^{3/2}-1\right)=4\sqrt{2}-2$

(b) $M=\int_C \delta\, ds=\int_0^1 (1)\left(2\sqrt{1+t^2}\right)dt=\left[t\sqrt{1+t^2}+\ln\left(t+\sqrt{1+t^2}\right)\right]_0^1=\left[\sqrt{2}+\ln\left(1+\sqrt{2}\right)\right]-(0+\ln 1)$

$=\sqrt{2}+\ln\left(1+\sqrt{2}\right)$

36. $\mathbf{r}(t)=t\mathbf{i}+2t\mathbf{j}+\frac{2}{3}t^{3/2}\mathbf{k},\ 0\le t\le 2 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{i}+2\mathbf{j}+t^{1/2}\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{1+4+t}=\sqrt{5+t};$

$M=\int_C \delta\, ds=\int_0^2\left(3\sqrt{5+t}\right)\left(\sqrt{5+t}\right)dt=\int_0^2 3(5+t)\,dt=\left[\frac{3}{2}(5+t)^2\right]_0^2=\frac{3}{2}\left(7^2-5^2\right)=\frac{3}{2}(24)=36;$

$M_{yz}=\int_C x\delta\, ds=\int_0^2 t[3(5+t)]\,dt=\int_0^2\left(15t+3t^2\right)dt=\left[\frac{15}{2}t^2+t^3\right]_0^2=30+8=38;$

$M_{xz}=\int_C y\delta\, ds=\int_0^2 2t[3(5+t)]\,dt=2\int_0^2\left(15t+3t^2\right)dt=76;\ M_{xy}=\int_C z\delta\, ds=\int_0^2 \frac{2}{3}t^{3/2}[3(5+t)]\,dt$

$=\int_0^2\left(10t^{3/2}+2t^{5/2}\right)dt=\left[4t^{5/2}+\frac{4}{7}t^{7/2}\right]_0^2=4(2)^{5/2}+\frac{4}{7}(2)^{7/2}=16\sqrt{2}+\frac{32}{7}\sqrt{2}=\frac{144}{7}\sqrt{2}$

$\Rightarrow \bar{x}=\frac{M_{yz}}{M}=\frac{38}{36}=\frac{19}{18},\ \bar{y}=\frac{M_{xz}}{M}=\frac{76}{36}=\frac{19}{9}$, and $\bar{z}=\frac{M_{xy}}{M}=\frac{144\sqrt{2}}{7\cdot 36}=\frac{4}{7}\sqrt{2}$

37. Let $x=a\cos t$ and $y=a\sin t,\ 0\le t\le 2\pi$. Then $\frac{dx}{dt}=-a\sin t,\ \frac{dy}{dt}=a\cos t,\ \frac{dz}{dt}=0$

$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=a\,dt;\ I_z=\int_C\left(x^2+y^2\right)\delta\, ds=\int_0^{2\pi}\left(a^2\sin^2 t+a^2\cos^2 t\right)a\delta\, dt$

$=\int_0^{2\pi}a^3\delta\, dt=2\pi\delta a^3.$

38. $\mathbf{r}(t)=t\mathbf{j}+(2-2t)\mathbf{k},\ 0\le t\le 1 \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{j}-2\mathbf{k} \Rightarrow \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{5};\ M=\int_C \delta\, ds=\int_0^1 \delta\sqrt{5}\,dt=\delta\sqrt{5};$

$I_x=\int_C\left(y^2+z^2\right)\delta\, ds=\int_0^1\left[t^2+(2-2t)^2\right]\delta\sqrt{5}\,dt=\int_0^1\left(5t^2-8t+4\right)\delta\sqrt{5}\,dt=\delta\sqrt{5}\left[\frac{5}{3}t^3-4t^2+4t\right]_0^1=\frac{5}{3}\delta\sqrt{5};$